1. Determine the general solution of \( \frac{\partial u}{\partial y} = 4ty \)

Since \( \frac{\partial u}{\partial y} = 4ty \) then 
\[
    u = \int 4ty \, dy = (4t) \int y \, dy \\
    = (4t) \frac{y^2}{2} + f(t) = 2ty^2 + f(t)
\]
i.e. 
\[
    u = 2ty^2 + f(t)
\]

2. Solve \( \frac{\partial u}{\partial t} = 2t \cos \theta \) given that \( u = 2t \) when \( \theta = 0 \)

Since \( \frac{\partial u}{\partial t} = 2t \cos \theta \) then 
\[
    u = \int 2t \cos \theta \, dt = (2 \cos \theta) \int t \, dt \\
    = (2 \cos \theta) \frac{t^2}{2} + f(\theta) = t^2 \cos \theta + f(\theta)
\]

\( u = 2t \) when \( \theta = 0 \), hence, 
\[
    2t = t^2 + f(\theta)
\]

from which, 
\[
    f(\theta) = 2t - t^2
\]

Hence, 
\[
    u = t^2 \cos \theta + 2t - t^2
\]
or 
\[
    u = t^2 (\cos \theta - 1) + 2t
\]

3. Verify that \( u(\theta, t) = \theta^2 + \theta t \) is a solution of \( \frac{\partial u}{\partial \theta} - 2 \frac{\partial u}{\partial t} = t \)

\[
    \frac{\partial u}{\partial \theta} = 2\theta + t \quad \text{and} \quad \frac{\partial u}{\partial t} = 0 + \theta = \theta
\]

Hence, 
\[
    \frac{\partial u}{\partial \theta} - 2 \frac{\partial u}{\partial t} = (2\theta + t) - 2(\theta) = 2\theta + t - 2\theta = t
\]
which verifies that \( \frac{\partial u}{\partial \theta} - 2 \frac{\partial u}{\partial t} = t \)
4. Verify that \( u = e^{-y} \cos x \) is a solution of \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)

Since \( u = e^{-y} \cos x \) then \( \frac{\partial u}{\partial x} = e^{-y} (-\sin x) \) and \( \frac{\partial^2 u}{\partial x^2} = -e^{-y} \cos x \)

Also, \( \frac{\partial u}{\partial y} = -e^{-y} \cos x \) and \( \frac{\partial^2 u}{\partial x^2} = e^{-y} \cos x \)

Hence, \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{-y} \cos x + e^{-y} \cos x = 0 \)

5. Solve \( \frac{\partial^2 u}{\partial x \partial y} = 8 e^y \sin 2x \) given that at \( y = 0 \), \( \frac{\partial u}{\partial x} = \sin x \), and at \( x = \frac{\pi}{2} \), \( u = 2 y^2 \)

Since \( \frac{\partial^2 u}{\partial x \partial y} = 8 e^y \sin 2x \) then integrating partially with respect to \( y \) gives:

\[
\frac{\partial u}{\partial x} = \int 8 e^y \sin 2x \, dy = 8 e^y \sin 2x + f(x)
\]

From the boundary conditions, \( \frac{\partial u}{\partial x} = \sin x \) when \( y = 0 \), hence

\[
\sin x = 8 e^0 \sin 2x + f(x) \quad \text{from which, } \quad f(x) = \sin x - 8 \sin 2x
\]

i.e. \( \frac{\partial u}{\partial x} = 8 e^y \sin 2x + \sin x - 8 \sin 2x \)

Integrating partially with respect to \( x \) gives:

\[
u = \int [8 e^y \sin 2x + \sin x - 8 \sin 2x] \, dx = -4 e^y \cos 2x - \cos x + 4 \cos 2x + f(y)
\]

From the boundary conditions, \( u = 2 y^2 \) when \( x = \frac{\pi}{2} \), hence

\[
2 y^2 = -4 e^y \cos \pi - 0 + 4 \cos \pi + f(y)
\]

\[
= 4 e^y - 4 + f(y) \quad \text{from which, } \quad f(y) = 2 y^2 - 4 e^y + 4
\]

Hence, the solution of \( \frac{\partial^2 u}{\partial x \partial y} = 8 e^y \sin 2x \) is given by:

\[
u = -4 e^y \cos 2x - \cos x + 4 \cos 2x + 2 y^2 - 4 e^y + 4
\]
6. Solve \( \frac{\partial^2 u}{\partial x^2} = y(4x^2 - 1) \) given that at \( x = 0 \), \( u = \sin y \) and \( \frac{\partial u}{\partial x} = \cos 2y \\

Since \( \frac{\partial^2 u}{\partial x^2} = y(4x^2 - 1) \) then \( \frac{\partial u}{\partial x} = \int y(4x^2 - 1) \, dx = y\left(\frac{4x^3}{3} - x\right) + f(y) \)

\( x = 0 \) when \( \frac{\partial u}{\partial x} = \cos 2y \), hence, \( \cos 2y = 0 + f(y) \)

Hence, \( \frac{\partial u}{\partial x} = y\left(\frac{4x^3}{3} - x\right) + \cos 2y \)

and \( u = \int \left[y\left(\frac{4x^3}{3} - x\right) + \cos 2y\right] \, dx = y\left(\frac{x^4}{3} - \frac{x^2}{2}\right) + x \cos 2y + F(y) \)

\( x = 0 \) when \( u = \sin y \), hence, \( \sin y = F(y) \)

Thus, \( u = y\left(\frac{x^4}{3} - \frac{x^2}{2}\right) + x \cos 2y + \sin y \)

7. Solve \( \frac{\partial^2 u}{\partial x \partial t} = \sin(x + t) \) given that \( \frac{\partial u}{\partial x} = 1 \) when \( t = 0 \), and when \( u = 2t \) when \( x = 0 \)

Since \( \frac{\partial^2 u}{\partial x \partial t} = \sin(x + t) \) then integrating partially with respect to \( t \) gives:

\( \frac{\partial u}{\partial x} = \int \sin(x + t) \, dt = -\cos(x + t) + f(x) \)

From the boundary conditions, \( \frac{\partial u}{\partial x} = 1 \) when \( t = 0 \), hence

\( 1 = -\cos x + f(x) \) from which, \( f(x) = 1 + \cos x \)

i.e. \( \frac{\partial u}{\partial x} = -\cos(x + t) + 1 + \cos x \)

Integrating partially with respect to \( x \) gives:

\( u = \int \left[-\cos(x + t) + 1 + \cos x\right] \, dx = -\sin(x + t) + x + \sin x + f(t) \)

From the boundary conditions, \( u = 2t \) when \( x = 0 \), hence

\( 2t = -\sin t + 0 + \sin 0 + f(t) = -\sin t + f(t) \) from which, \( f(t) = 2t + \sin t \)
Hence, the solution of \(\frac{\partial^2 u}{\partial x \partial t} = \sin(x+t)\) is given by:

\[ u = -\sin(x+t) + x + \sin x + 2t + \sin t \]

8. Show that \(u(x, y) = xy + \frac{x}{y}\) is a solution of \(2x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = 2x\)

Since \(u = xy + \frac{x}{y}\) then \(\frac{\partial u}{\partial x} = y + \frac{1}{y}\) and \(\frac{\partial u}{\partial y} = x - \frac{x}{y^2} = x - xy^{-2}\)

and \(\frac{\partial^2 u}{\partial y^2} = \frac{2x}{y^3}\)

Also, \(\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(x - \frac{x}{y^2}\right) = 1 - \frac{1}{y^2}\)

Hence, \(L.H.S. = 2x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = 2x \left(1 - \frac{1}{y^2}\right) + y \left(\frac{2x}{y^3}\right)\)

\[= 2x - \frac{2x}{y^2} + \frac{2x}{y^2} = 2x = R.H.S.\]

9. Find the particular solution of the differential equation \(\frac{\partial^2 y}{\partial x \partial y} = \cos x \cos y\) given the initial conditions that when \(y = \pi\), \(\frac{\partial u}{\partial x} = x\), and when \(x = \pi\), \(u = 2 \cos y\).

Since \(\frac{\partial^2 y}{\partial x \partial y} = \cos x \cos y\) then integrating partially with respect to \(y\) gives:

\[\frac{\partial u}{\partial x} = \int \cos x \cos y \, dy = \cos x \sin y + f(x)\]

From the boundary conditions, \(\frac{\partial u}{\partial x} = x\) when \(y = \pi\), hence

\[x = \cos x \sin \pi + f(x)\]

from which, \(f(x) = x\)

i.e. \(\frac{\partial u}{\partial x} = \cos x \sin y + x\)

Integrating partially with respect to \(x\) gives:
\[ u = \int [\cos x \sin y + x] \, dx = \sin x \sin y + \frac{x^2}{2} + f(y) \]

From the boundary conditions, \( u = 2 \cos y \) when \( x = \pi \), hence

\[
2 \cos y = \sin \pi \sin y + \frac{\pi^2}{2} + f(y)
\]

\[
= \frac{\pi^2}{2} + f(y) \quad \text{from which, } \quad f(y) = 2 \cos y - \frac{\pi^2}{2}
\]

Hence, the solution of \( \frac{\partial^2 y}{\partial x \partial y} = \cos x \cos y \) is given by:

\[
u = \sin x \sin y + \frac{x^2}{2} + 2 \cos y - \frac{\pi^2}{2}
\]

**10.** Verify that \( \phi(x, y) = x \cos y + e^{x} \sin y \) satisfies the differential equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + x \cos y = 0
\]

Since \( \phi = x \cos y + e^{x} \sin y \) then \( \frac{\partial \phi}{\partial x} = \cos y + e^{x} \sin y \) and \( \frac{\partial^2 \phi}{\partial x^2} = e^{x} \sin y \)

and

\[
\frac{\partial \phi}{\partial y} = -x \sin y + e^{x} \cos y \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = -x \cos y - e^{x} \sin y
\]

Hence, \( \text{L.H.S.} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + x \cos y = e^{x} \sin y + (x \cos y) + x \cos y = e^{x} \sin y - x \cos y - e^{x} \sin y + x \cos y = 0 = \text{R.H.S.} \)
EXERCISE 319 Page 888

1. Solve $T'' = c^2 \mu T$ given $c = 3$ and $\mu = 1$

Since $T'' = c^2 \mu T$ and $c = 3$ and $\mu = 1$, then $T'' - (3)^2(1)T = 0$ i.e. $T'' - 9T = 0$

If $T'' - 9T = 0$ then the auxiliary equation is:

$$m^2 - 9 = 0 \quad \text{i.e.} \quad m^2 = 9 \quad \text{from which,} \quad m = \sqrt{9} = \pm 3$$

Thus, the general solution is: $T = Ae^{3t} + Be^{-3t}$

2. Solve $T'' - c^2 \mu T = 0$ given $c = 3$ and $\mu = -1$

Since $T'' - c^2 \mu T = 0$ and $c = 3$ and $\mu = -1$, then $T'' - (3)^2(-1)T = 0$ i.e. $T'' + 9T = 0$

If $T'' + 9T = 0$ then the auxiliary equation is:

$$m^2 + 9 = 0 \quad \text{i.e.} \quad m^2 = -9 \quad \text{from which,} \quad m = \sqrt{-9} = \pm 3j \quad \text{or} \quad 0 \pm 3j$$

Thus, the general solution is: $T = e^t \{A \cos 3t + B \sin 3t\}$

$$= A \cos 3t + B \sin 3t$$

3. Solve $X'' = \mu X$ given $\mu = 1$

Since $X'' = \mu X$ and $\mu = 1$, then $X'' - X = 0$

If $X'' - X = 0$ then the auxiliary equation is:

$$m^2 - 1 = 0 \quad \text{i.e.} \quad m^2 = 1 \quad \text{from which,} \quad m = 1 \quad \text{or} \quad m = -1$$

Thus, the general solution is: $X = Ae^t + Be^{-t}$

4. Solve $X'' - \mu X = 0$ given $\mu = -1$

Since $X'' - \mu X = 0$ and $\mu = -1$, then $X'' + X = 0$

If $X'' + X = 0$ then the auxiliary equation is:
\[ m^2 + 1 = 0 \quad \text{i.e.} \quad m^2 = -1 \quad \text{from which,} \quad m = \sqrt{-1} = \pm j \]

Thus, the general solution is:

\[ X = e^0 \left\{ A \cos x + B \sin x \right\} \]

\[ = A \cos x + B \sin x \]
1. An elastic string is stretched between two points 40 cm apart. Its centre point is displaced 1.5 cm from its position of rest at right-angles to the original direction of the string and then released with zero velocity. Determine the subsequent motion $u(x, t)$ by applying the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{with} \quad c^2 = 9$$

The elastic string is shown in the diagram below

\[ u(x, 0) = f(x) = \frac{1.5}{20} x \quad 0 \leq x \leq 20 \]

\[ = -\frac{1.5}{20} x + 3 = \frac{60 - 1.5x}{20} \quad 20 \leq x \leq 40 \]

\[ \left[ \frac{\partial u}{\partial t} \right]_{t=0} = 0 \quad \text{i.e. zero initial velocity} \]

2. Assuming a solution $u = XT$, where $X$ is a function of $x$ only, and $T$ is a function of $t$ only,

then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial^2 u}{\partial x^2} = X''T$ and $\frac{\partial u}{\partial y} = XT'$ and $\frac{\partial^2 u}{\partial y^2} = XT''$

Substituting into the partial differential equation, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

gives: $X''T = \frac{1}{c^2} XT''$ i.e. $X''T = \frac{1}{9} XT''$ since $c^2 = 9$

3. Separating the variables gives:

$$\frac{X''}{X} = \frac{T''}{9T}$$
Let constant, \( \mu = \frac{X''}{X} = \frac{T''}{9T} \) then \( \mu = \frac{X''}{X} \) and \( \mu = \frac{T''}{9T} \)

from which, \( X'' - \mu X = 0 \) and \( T'' - 9\mu T = 0 \)

4. Letting \( \mu = -p^2 \) to give an oscillatory solution gives

\[ X'' + p^2 X = 0 \] 

and the auxiliary equation is: \( m^2 + p^2 = 0 \) from which, \( m = \sqrt{-p^2} = \pm jp \)

and \( T'' + 9p^2 T = 0 \) and the auxiliary equation is:

\[ m^2 + 9p^2 = 0 \] 

from which, \( m = \sqrt{-9p^2} = \pm 3jp \)

5. Solving each equation gives: \( X = A \cos px + B \sin px \) and \( T = C \cos 3pt + D \sin 3pt \)

Thus, \( u(x,t) = \{A \cos px + B \sin px\} \{C \cos 3pt + D \sin 3pt\} \)

6. Applying the boundary conditions to determine constants \( A \) and \( B \) gives:

(i) \( u(0,t) = 0 \), hence \( 0 = A \{C \cos 3pt + D \sin 3pt\} \) from which we conclude that \( A = 0 \)

Therefore, \( u(x,t) = B \sin px \{C \cos 3pt + D \sin 3pt\} \) \hspace{1cm} (1)

(ii) \( u(40,t) = 0 \), hence \( 0 = B \sin 40p \{C \cos 3pt + D \sin 3pt\} \)

\( B \neq 0 \) hence \( \sin 40p = 0 \) from which, \( 40p = n\pi \) and \( p = \frac{n\pi}{40} \)

7. Substituting in equation (1) gives: \( u(x,t) = B \sin \frac{n\pi x}{40} \left\{C \cos \frac{3n\pi t}{40} + D \sin \frac{3n\pi t}{40}\right\} \)

or, more generally, \( u_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{40} \left\{A_n \cos \frac{3n\pi t}{40} + B_n \sin \frac{3n\pi t}{40}\right\} \) \hspace{1cm} (2)

where \( A_n = BC \) and \( B_n = BD \)

8. From equation (8), page 890 of textbook,

\[ A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \]

\[ = \frac{2}{40} \left[ \int_0^{20} \left( 1.5 - 1.5x \right) \sin \frac{n\pi x}{40} \, dx + \int_0^{40} \left( 60 - 1.5x \right) \sin \frac{n\pi x}{40} \, dx \right] \]

Each integral is determined using integration by parts (see Chapter 68, page 739) with the result:

\[ A_n = \frac{(8)(1.5)}{n^2 \pi^2} \sin \frac{n\pi}{2} = \frac{12}{n^2 \pi^2} \sin \frac{n\pi}{2} \]
From equation (9), page 890 of textbook, \( B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \)

\[
\left[ \frac{\partial u}{\partial t} \right]_{t=0} = 0 = g(x) \quad \text{thus,} \quad B_n = 0
\]

Substituting into equation (2) gives:

\[
u_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{40} \left\{ A_n \cos \frac{3n\pi t}{40} + B_n \sin \frac{3n\pi t}{40} \right\}
\]

\[
= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{40} \left\{ \frac{12}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{3n\pi t}{40} + (0) \sin \frac{n\pi t}{50} \right\}
\]

Hence,

\[
u(x, t) = \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{2} \sin \frac{n\pi x}{40} \cos \frac{3n\pi t}{40}
\]

2. The centre point of an elastic string between two points \( P \) and \( Q \), 80 cm apart, is deflected a distance of 1 cm from its position of rest perpendicular to \( PQ \) and released initially with zero velocity. Apply the wave equation \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \) where \( c = 8 \), to determine the motion of a point distance \( x \) from \( P \) at time \( t \).

The elastic string is shown in the diagram below.

The boundary and initial conditions given are:

\[
u(0, t) = 0 \quad \begin{cases} \vspace{1em} 
\nu(80, t) = 0
\end{cases}
\]

\[
u(x, 0) = f(x) = \begin{cases} \vspace{1em}
\frac{1}{40} x & \text{if} \quad 0 \leq x \leq 40 \\
-\frac{1}{40} x + 2 & \text{if} \quad 40 \leq x \leq 80
\end{cases}
\]
\[
\left[ \frac{\partial u}{\partial t} \right]_{t=0} = 0 \quad \text{i.e. zero initial velocity}
\]

Assuming a solution \( u = XT \), where \( X \) is a function of \( x \) only, and \( T \) is a function of \( t \) only,
then \( \frac{\partial u}{\partial x} = X'T \) and \( \frac{\partial^2 u}{\partial x^2} = X'^2T \) and \( \frac{\partial u}{\partial y} = XT' \) and \( \frac{\partial^2 u}{\partial y^2} = XT'' \)

Substituting into the partial differential equation,
\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]
gives:
\[
X'^2T = \frac{1}{c^2} XT'' \quad \text{i.e.} \quad X'^2T = \frac{1}{64} XT'' \quad \text{since} \ c = 8
\]

Separating the variables gives:
\[
\frac{X''}{X} = \frac{T''}{64T}
\]

Let constant, \( \mu = \frac{X''}{X} = \frac{T''}{64T} \) then \( \mu = \frac{X''}{X} \) and \( \mu = \frac{T''}{64T} \)

from which,
\[
X'' - \mu X = 0 \quad \text{and} \quad T'' - 64 \mu T = 0
\]

Letting \( \mu = -p^2 \) to give an oscillatory solution gives
\[
X'' + p^2 X = 0 \quad \text{and the auxiliary equation is:} \quad m^2 + p^2 = 0 \quad \text{from which,} \quad m = \sqrt{-p^2} = \pm j p
\]
and \( T'' + 64 p^2 T = 0 \) \quad \text{and the auxiliary equation is:}
\[
m^2 + 64 p^2 = 0 \quad \text{from which,} \quad m = \sqrt{-64 p^2} = \pm j8 p
\]

Solving each equation gives:
\[
X = A \cos px + B \sin px \quad \text{and} \quad T = C \cos 8pt + D \sin 8pt
\]

Thus, \( u(x, t) = \{ A \cos px + B \sin px \} \{ C \cos 8pt + D \sin 8pt \} \)

Applying the boundary conditions to determine constants \( A \) and \( B \) gives:

(i) \( u(0, t) = 0 \), hence \( 0 = A \{ C \cos 8pt + D \sin 8pt \} \) from which we conclude that \( A = 0 \)

Therefore, \( u(x, t) = B \sin px \{ C \cos 8pt + D \sin 8pt \} \) \quad (1)

(ii) \( u(80, t) = 0 \), hence \( 0 = B \sin 80p \{ C \cos 8pt + D \sin 8pt \} \)

\( B \neq 0 \) hence \( \sin 80p = 0 \) from which, \( 80p = n\pi \) and \( p = \frac{n\pi}{80} \)

Substituting in equation (1) gives:
\[
u(x, t) = B \sin \frac{n\pi x}{80} \left\{ C \cos \frac{8n\pi t}{80} + D \sin \frac{8n\pi t}{80} \right\}
\]

or, more generally,
\[
u_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{80} \left\{ A_n \cos \frac{n\pi t}{10} + B_n \sin \frac{n\pi t}{10} \right\} \quad (2)
\]
\[ A_n = \frac{(8)(1)}{n^2\pi^2} \sin \frac{n\pi}{2} = \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \]

\[ B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \quad \left[ \frac{\partial u}{\partial t} \right]_{t=0} = 0 = g(x) \quad \text{thus, } B_n = 0 \]

Substituting into equation (2) gives:

\[ u_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{80} \left\{ A_n \cos \frac{n\pi t}{10} + B_n \sin \frac{n\pi t}{10} \right\} \]

\[ = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{80} \left\{ \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{n\pi t}{10} + (0) \sin \frac{n\pi t}{10} \right\} \]

Hence,

\[ u(x, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{2} \sin \frac{n\pi t}{80} \cos \frac{n\pi t}{10} \]
1. A metal bar, insulated along its sides, is 4 m long. It is initially at a temperature of 10°C and at

time \( t = 0 \), the ends are placed into ice at 0°C. Find an expression for the temperature at a point \( P \)
at a distance \( x \) m from one end at any time \( t \) seconds after \( t = 0 \)

The temperature \( u \) along the length of the bar is shown in the diagram below

The heat conduction equation is

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}
\]

and the given boundary conditions are:

\[ u(0, t) = 0, \quad u(4, t) = 0 \quad \text{and} \quad u(x, 0) = 10 \]

Assuming a solution of the form \( u = XT \), then, \( X = A \cos px + B \sin px \)

and

\[ T = k e^{-\frac{c^2}{4}t} \]

Thus, the general solution is given by:\n
\[ u(x, t) = \{P \cos px + Q \sin px\} e^{-\frac{c^2}{4}t} \]

\[ u(0, t) = 0 \quad \text{thus} \quad 0 = P e^{-\frac{c^2}{4}t} \quad \text{from which} \quad P = 0 \quad \text{and} \quad u(x, t) = \{Q \sin px\} e^{-\frac{c^2}{4}t} \]

Also, \( u(4, t) = 0 \quad \text{thus} \quad 0 = \{Q \sin 4p\} e^{-\frac{c^2}{4}t} \)

Since \( Q \neq 0 \), \( \sin 4p = 0 \) from which, \( 4p = n\pi \) where \( n = 1, 2, 3, \ldots \) and \( p = \frac{n\pi}{4} \)

Hence,

\[ u(x, t) = \sum_{n=1}^{\infty} \left\{Q_n e^{-\frac{c^2}{4}t} \sin \frac{n\pi x}{4}\right\} \]

The final initial condition given was that at \( t = 0, u = 10 \), i.e. \( u(x, 0) = f(x) = 10 \)

Hence,

\[ 10 = \sum_{n=1}^{\infty} \left\{Q_n \sin \frac{n\pi x}{4}\right\} \]
where, from Fourier coefficients, \( Q_n = 2 \times \text{mean value of } 10 \sin \frac{n\pi x}{4} \) from \( x = 0 \) to \( x = 4 \)

i.e. \( Q_n = \frac{2}{4} \int_0^4 10 \sin \frac{n\pi x}{4} \, dx = 5 \left[ -\frac{\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_0^4 = -\frac{20}{n\pi} \left[ \cos \frac{4n\pi}{4} - \cos 0 \right] = \frac{20}{n\pi} (1 - \cos n\pi) \)

\[= 0 \text{ (when } n \text{ is even) and } \frac{40}{n\pi} \text{ (when } n \text{ is odd)} \]

Hence, the required solution is: \( u(x, t) = \sum_{n=1}^\infty Q_n e^{-px^2/16} \sin \frac{n\pi x}{4} \)

\[= \frac{40}{\pi} \sum_{n=\text{odd}}^\infty \frac{1}{n} e^{-\frac{n^2\pi^2}{16}} \sin \frac{n\pi x}{4} \]

2. An insulated uniform metal bar, 8 m long, has the temperature of its ends maintained at 0°C, and at time \( t = 0 \) the temperature distribution \( f(x) \) along the bar is defined by \( f(x) = x(8 - x) \). If \( c^2 = 1 \), solve the heat conduction equation \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \) to determine the temperature \( u \) at any point in the bar at time \( t \).

The temperature \( u \) along the length of bar is shown in the diagram below

The heat conduction equation is \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \) and the given boundary conditions are:

\[u(0, t) = 0, \quad u(8, t) = 0 \quad \text{and} \quad u(x, 0) = 0\]

Assuming a solution of the form \( u = XT \), then, \( X = A \cos px + B \sin px \)

and \( T = k e^{-px^2t} \)

Thus, the general solution is given by: \( u(x, t) = \{ P \cos px + Q \sin px \} e^{-px^2t} \)

\( u(0, t) = 0 \) thus \( 0 = P e^{-p^2t} \) from which, \( P = 0 \) and \( u(x, t) = \{ Q \sin px \} e^{-px^2t} \)

Also, \( u(8, t) = 0 \) thus \( 0 = \{ Q \sin 8p \} e^{-8px^2t} \)
Since $Q \neq 0$, $\sin 8p = 0$ from which, $8p = n\pi$ where $n = 1, 2, 3, \ldots$ and $p = \frac{n\pi}{8}$

Hence, \[ u(x, t) = \sum_{n=1}^{\infty} \left\{ Q_n e^{-p^2ct} \sin \frac{n\pi x}{8} \right\} \]

The final initial condition given was that at $t = 0$, $u = 0$, i.e. $u(x, 0) = f(x) = 0$

Hence, \[ 0 = \sum_{n=1}^{\infty} \left\{ Q_n \sin \frac{n\pi x}{8} \right\} \]

where, from Fourier coefficients, $Q_n = 2 \times$ mean value of $x(8 - x) \sin \frac{n\pi x}{8}$ from $x = 0$ to $x = 8$,

i.e. \[ Q_n = \frac{2}{8} \int_{0}^{8} x(8 - x) \sin \frac{n\pi x}{8} \, dx = \frac{1}{4} \left\{ \int_{0}^{8} 8x \sin \frac{n\pi x}{8} \, dx - \int_{0}^{8} x^2 \sin \frac{n\pi x}{8} \, dx \right\} \]

Using integration by parts, \[ \frac{1}{4} \int_{0}^{8} 8x \sin \frac{n\pi x}{8} \, dx = \left( \frac{16}{n\pi} \right) \cos n\pi + \left( \frac{16}{(n\pi)^2} \right) \sin n\pi \]

and

\[ \frac{1}{4} \int_{0}^{8} x^2 \sin \frac{n\pi x}{8} \, dx = \left( \frac{2}{n\pi} \right)^2 \cos n\pi - \left( \frac{4}{(n\pi)^2} \right) \sin n\pi \]

\[ = 0 \text{ (when } n \text{ is even)} \quad \text{and} \quad \left( \frac{8}{\pi} \right)^3 \text{ (when } n \text{ is odd)} \]

Hence, the required solution is: \[ u(x, t) = \sum_{n=1}^{\infty} \left\{ Q_n e^{-p^2ct} \sin \frac{n\pi x}{8} \right\} = \sum_{n=1}^{\infty} Q_n e^{-p^2 \left( \frac{n\pi}{8} \right)^2 t} \sin \frac{n\pi x}{8} \]

\[ = \left( \frac{8}{\pi} \right)^3 \sum_{n(odd)}^{\infty} \frac{1}{n^3} e^{-\frac{n^2\pi^2 t}{64}} \sin \frac{n\pi x}{8} \]

3. The ends of an insulated rod $PQ$, 20 units long, are maintained at 0°C. At time $t = 0$, the temperature within the rod rises uniformly from each end reaching 4°C at the mid-point of $PQ$.

Find an expression for the temperature $u(x, t)$ at any point in the rod, distant $x$ from $P$ at any time $t$ after $t = 0$. Assume the heat conduction equation to be \[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \] and take $c^2 = 1$

The temperature along the length of the rod is shown in the diagram below.
The heat conduction equation is \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \) and the given boundary conditions are:

\[ u(0, t) = 0, \quad u(20, t) = 0 \quad \text{and} \quad u(x, 0) = 0 \]

Assuming a solution of the form \( u = XT \), then,

\[ X = A \cos px + B \sin px \]

and \( T = k e^{-\nu^2 c t} \)

Thus, the general solution is given by: \( u(x, t) = \{ P \cos px + Q \sin px \} e^{-\nu^2 c t} \)

\[ u(0, t) = 0 \quad \text{thus} \quad 0 = P e^{-\nu^2 c t} \quad \text{from which,} \quad P = 0 \quad \text{and} \quad u(x, t) = \{ Q \sin px \} e^{-\nu^2 c t} \]

Also, \( u(20, t) = 0 \quad \text{thus} \quad 0 = \{ Q \sin 20p \} e^{-\nu^2 c t} \)

Since \( Q \neq 0, \sin 20p = 0 \) from which, \( 20p = n\pi \) where \( n = 1, 2, 3, \ldots \) and \( p = \frac{n\pi}{20} \)

Hence, \( u(x, t) = \sum_{n=1}^{\infty} \{ Q_n e^{-\nu^2 c t} \sin \frac{n\pi x}{20} \} \)

where, from Fourier coefficients, \( 2 \times \) the mean value from \( x = 0 \) to \( x = 20 \)

\[ Q_n = \frac{2}{20} \left[ \int_0^{10} \left( \frac{2}{5} x \right) \sin \frac{n\pi x}{20} \, dx + \int_{10}^{20} \left( \frac{2}{5} x - 8 \right) \sin \frac{n\pi x}{20} \, dx \right] \quad \text{(see above diagram)} \]

\[ = \frac{1}{10} \left\{ \left( \frac{2}{5} \right) \cos \frac{n\pi x}{20} + \frac{2}{5} \sin \frac{n\pi x}{20} \right\}^{10}_0 \left[ \left( \frac{2}{5} \right) \cos \frac{n\pi x}{20} + \frac{2}{5} \sin \frac{n\pi x}{20} \right]^{20}_{10} - \left( \frac{2}{5} \right) \cos \frac{n\pi x}{20} + \frac{2}{5} \sin \frac{n\pi x}{20} \right\}^{20}_0 \]

\[ = \frac{1}{10} \left[ \left( \frac{-4 \cos \frac{n\pi}{2}}{n\pi} + \frac{4 \sin \frac{n\pi}{2}}{n\pi^2} \right) - (0) \right] + \left[ \left( \frac{8 \cos \frac{n\pi}{2}}{n\pi} + 0 - 8 \cos \frac{n\pi}{2} \right) - \left( \frac{4 \cos \frac{n\pi}{2}}{n\pi} - \frac{4 \sin \frac{n\pi}{2}}{n\pi^2} - \frac{8 \cos \frac{n\pi}{2}}{20} \right) \right] \]
\[
\begin{align*}
&= \frac{1}{10} \left\{ -4 \cos \frac{n\pi}{2} \left( \frac{n\pi}{20} \right)^2 + 8 \cos n\pi \left( \frac{n\pi}{20} \right) - 8 \cos \frac{n\pi}{2} \left( \frac{n\pi}{20} \right)^2 + 8 \cos n\pi \left( \frac{n\pi}{20} \right)^2 \right\} \\
&= \frac{1}{10} \left\{ -8 \cos n\pi \left( \frac{n\pi}{20} \right)^2 + 8 \sin n\pi \left( \frac{n\pi}{20} \right) \right\} = \frac{1}{10} \left\{ 8 \sin n\pi \left( \frac{n\pi}{20} \right)^2 \right\} \\
&= 0 \text{ when } n \text{ is even} \\
&= \frac{8}{10} \left( \frac{20}{n\pi} \right)^2 \sin \frac{n\pi}{2} = \frac{320}{n^2 \pi^2} \sin \frac{n\pi}{2} \text{ when } n \text{ is odd}
\end{align*}
\]

Hence, the required solution is:

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{320}{n^2 \pi^2} \sin \frac{n\pi}{2} e^{-\frac{\pi^2 n^2 (1)^2}{200}} \sin \frac{n\pi x}{20} \\
= \frac{320}{\pi^2} \sum_{n(\text{odd})=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{20} e^{-\left( \frac{n^2 \pi^2 i}{400} \right)}
\end{align*}
\]
1. A rectangular plate is bounded by the lines \( x = 0, y = 0, x = 1 \) and \( y = 3 \). Apply the Laplace equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) to determine the potential distribution \( u(x, y) \) over the plate, subject to the following boundary conditions:

\[
\begin{align*}
&u = 0 \text{ when } x = 0 \quad 0 \leq y \leq 2,
&u = 0 \text{ when } x = 1 \quad 0 \leq y \leq 2,
&u = 0 \text{ when } y = 2 \quad 0 \leq x \leq 1,
&u = 5 \text{ when } y = 3 \quad 0 \leq x \leq 1
\end{align*}
\]

Initially a solution of the form \( u(x, y) = X(x)Y(y) \) is assumed, where \( X \) is a function of \( x \) only, and \( Y \) is a function of \( y \) only. Simplifying to \( u = XY \), determining partial derivatives, and substituting into \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) gives:

\[
X''Y + XY'' = 0
\]

Separating the variables gives:

\[
\frac{X''}{X} = -\frac{Y''}{Y}
\]

Letting each side equal a constant, \( -p^2 \) gives the two equations:

\[
X'' + p^2 X = 0 \quad \text{and} \quad Y'' - p^2 Y = 0
\]

from which, \( X = A \cos px + B \sin px \)

and \( Y = C e^{py} + D e^{-py} \) or \( Y = C \cosh py + D \sinh py \) or \( Y = E \sinh p(y + \phi) \)

Hence \( u(x, y) = XY = \{A \cos px + B \sin px \} \{E \sinh p(y + \phi)\} \)

or \( u(x, y) = \{P \cos px + Q \sin px \} \{\sinh p(y + \phi)\} \) where \( P = AE \) and \( Q = BE \)

The first boundary condition is: \( u(0, y) = 0 \), hence \( 0 = P \sinh p(y + \phi) \) from which, \( P = 0 \)

Hence, \( u(x, y) = Q \sin px \sinh p(y + \phi) \)

The second boundary condition is: \( u(1, y) = 0 \), hence \( 0 = Q \sin p(1) \sinh p(y + \phi) \)

from which, \( \sin p = 0 \), hence, \( p = n\pi \) for \( n = 1, 2, 3, \ldots \)

The third boundary condition is: \( u(x, 2) = 0 \), hence \( 0 = Q \sin px \sinh p(2 + \phi) \)

from which, \( \sinh p(2 + \phi) = 0 \) and \( \phi = -2 \)

Hence, \( u(x, y) = Q \sin px \sinh p(y - 2) \)

Since there are many solutions for integer values of \( n \)

\[
u(x, y) = \sum_{n=1}^{\infty} Q_n \sin px \sinh p(y - 2) = \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi(y - 2) \quad (a)
\]
The fourth boundary condition is: \( u(x, 3) = 5 = f(x) \), hence, \( f(x) = \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi(3 - 2) \)

From Fourier series coefficients,

\[ Q_n \sinh n\pi = 2 \times \text{the mean value of } f(x) \sin n\pi x \text{ from } x = 0 \text{ to } x = 1 \]

i.e.

\[ = \frac{2}{1} \int_{0}^{1} 5 \sin n\pi x \, dx = 10 \left[ -\frac{\cos n\pi x}{n\pi} \right]_{0}^{1} = -\frac{10}{n\pi} (\cos n\pi - \cos 0) = \frac{10}{n\pi} (1 - \cos n\pi) \]

\[ = 0 \text{ (for even values of } n) \]

\[ = \frac{20}{n\pi} \text{ (for odd values of } n) \]

Hence, \( Q_n = \frac{20}{n\pi (\sinh n\pi)} = \frac{20}{n\pi} \cosh n\pi \)

Hence, from equation (a), \( u(x, y) = \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi(y - 2) \)

\[ = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cosh n\pi \sin n\pi x \sinh n\pi(y - 2) \]

\[ \]

2. A rectangular plate is bounded by the lines \( x = 0, y = 0, x = 3, y = 2 \). Determine the potential distribution \( u(x, y) \) over the rectangle using the Laplace equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \), subject to the following boundary conditions: \( u(0, y) = 0 \quad 0 \leq y \leq 2, \quad u(3, y) = 0 \quad 0 \leq y \leq 2, \quad u(x, 2) = 0 \quad 0 \leq x \leq 3, \quad u(x, 0) = x(3 - x) \quad 0 \leq x \leq 3 \)

Initially a solution of the form \( u(x, y) = X(x)Y(y) \) is assumed, where \( X \) is a function of \( x \) only, and \( Y \) is a function of \( y \) only. Simplifying to \( u = XY \), determining partial derivatives, and substituting into \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) gives:

\[ X''Y + XY'' = 0 \]

Separating the variables gives:

\[ \frac{X''}{X} = -\frac{Y''}{Y} \]

Letting each side equal a constant, \(-p^2\) gives the two equations:

\[ X'' + p^2 X = 0 \quad \text{and} \quad Y'' - p^2 Y = 0 \]

from which, \( X = A \cos px + B \sin px \)

and \( Y = C e^{py} + D e^{-py} \) or \( Y = C \cosh py + D \sinh py \) or \( Y = E \sinh p(y + \phi) \)

Hence \( u(x, y) = XY = \{A \cos px + B \sin px\} \{E \sinh p(y + \phi)\} \)

or \( u(x, y) = \{P \cos px + Q \sin px\} \{\sinh p(y + \phi)\} \)

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The first boundary condition is: \( u(0, y) = 0 \), hence \( 0 = P \sinh p(y + \phi) \) from which, \( P = 0 \)

Hence, \( u(x, y) = Q \sin px \sinh p(y + \phi) \)

The second boundary condition is: \( u(3, y) = 0 \), hence \( 0 = Q \sin 3p \sinh p(y + \phi) \)

from which, \( \sin 3p = 0 \), hence, \( 3p = n\pi \) i.e. \( p = \frac{n\pi}{3} \) for \( n = 1, 2, 3, \ldots \)

The third boundary condition is: \( u(x, 2) = 0 \), hence, \( 0 = Q \sin px \sinh p(2 + \phi) \)

from which, \( \sinh p(2 + \phi) = 0 \) and \( \phi = -2 \)

Hence, \( u(x, y) = Q \sin px \sinh p(y - 2) \)

Since there are many solutions for integer values of \( n \),

\[
   u(x, y) = \sum_{n=1}^{\infty} Q_n \sin px \sinh p(y - 2) = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi}{3} x \sinh \frac{n\pi}{3} (y - 2) \quad (a)
\]

The fourth boundary condition is: \( u(x, 0) = x(3 - x) = 3x - x^2 = f(x) \),

hence,

\[
   f(x) = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi}{3} x \sinh \frac{n\pi}{3} (-2)
\]

From Fourier series coefficients,

\[
   Q_n \sinh \frac{-2n\pi}{3} = 2 \times \text{the mean value of } f(x) \sin \frac{n\pi}{3} x \text{ from } x = 0 \text{ to } x = 3
\]

\[
   = \frac{2}{1} \int_0^3 (3x - x^2) \sin \frac{n\pi}{3} x \, dx
\]

\[
   = 2 \left\{ \left[ -3x \cos \frac{n\pi x}{3} + 3\sin \frac{n\pi x}{3} \right]_0^3 - \left[ -x^2 \cos \frac{n\pi x}{3} + 2x \sin \frac{n\pi x}{3} + 2 \cos \frac{n\pi x}{3} \right]_0^3 \right\}
\]

by integration by parts (see Chapter 68)

\[
   = 2 \left\{ -9\cos n\pi + 9\cos n\pi - \frac{2\cos n\pi}{3} + \frac{2\cos n\pi}{3} \right\} = 2 \left\{ \frac{27}{n^3\pi^3} \left( 2 - 2\cos n\pi \right) \right\}
\]

\[
   = \frac{54}{n^3\pi^3} (2 - 2\cos n\pi)
\]

\[
   = 0 \text{ (for even values of } n), \quad = \frac{216}{n^3\pi^3} \text{ (for odd values of } n)
\]
Hence, 
\[ Q_n = \frac{216}{n^3\pi^3} \left( \sinh \frac{-2n\pi}{3} \right) = \frac{216}{n^3\pi^3} \cosec \frac{-2n\pi}{3} \]

Hence, from equation (a),
\[ u(x, y) = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi}{3} x \sinh \frac{n\pi}{3} (y - 2) = \sum_{n=1}^{\infty} \frac{216}{n^3\pi^3} \cosec \frac{-2n\pi}{3} \sin \frac{n\pi x}{3} \sinh \frac{n\pi}{3} (y - 2) \]
\[ = \frac{216}{\pi^3} \sum_{n \ (\text{odd})=1}^{\infty} \frac{1}{n^3} \cosec \frac{2n\pi}{3} \sin \frac{n\pi x}{3} \sinh \frac{n\pi}{3} (2 - y) \]