CHAPTER 30 IRREGULAR AREAS AND VOLUMES AND MEAN VALUES

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1. Plot a graph of \( y = 3x - x^2 \) by completing a table of values of \( y \) from \( x = 0 \) to \( x = 3 \). Determine the area enclosed by the curve, the \( x \)-axis and ordinate \( x = 0 \) and \( x = 3 \) by (a) the trapezoidal rule, (b) the mid-ordinate rule and (c) by Simpson’s rule.

A table of values is shown below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 0.5 )</th>
<th>( 1.0 )</th>
<th>( 1.5 )</th>
<th>( 2.0 )</th>
<th>( 2.5 )</th>
<th>( 3.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 3x - x^2 )</td>
<td>( 0 )</td>
<td>( 1.25 )</td>
<td>( 2.0 )</td>
<td>( 2.25 )</td>
<td>( 2.0 )</td>
<td>( 1.25 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

A graph of \( y = 3x - x^2 \) is shown below.

(a) Using the trapezoidal rule, with six intervals each of width 0.5 gives:

\[
\text{area} \approx (0.5) \left[ \frac{0 + 0}{2} + 1.25 + 2.0 + 2.25 + 2.0 + 1.25 \right] = (0.5)(8.75) = 4.375 \text{ square units}
\]

(b) Using the mid-ordinate rule, with six intervals, with mid-ordinates occurring at

<table>
<thead>
<tr>
<th>( 0.25 )</th>
<th>( 0.75 )</th>
<th>( 1.25 )</th>
<th>( 1.75 )</th>
<th>( 2.25 )</th>
<th>( 2.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.6875 )</td>
<td>( 1.6875 )</td>
<td>( 2.1875 )</td>
<td>( 2.1875 )</td>
<td>( 1.6875 )</td>
<td>( 0.6875 )</td>
</tr>
</tbody>
</table>

\[
\text{area} \approx (0.5)[0.6875 + 1.6875 + 2.1875 + 2.1875 + 1.6875 + 0.6875] = (0.5)(9.125)
\]

\[
= 4.563 \text{ square units}
\]
(c) Using Simpson’s rule, with six intervals each of width 0.5 gives:

\[
\text{area} = \frac{1}{3} (0.5) [(0 + 0) + 4(1.25 + 2.25 + 1.25) + 2(2.0 + 2.0)] = \frac{1}{3} (0.5) [0 + 19 + 8] \\
= \frac{1}{3} (0.5)(27) = 4.5 \text{ square units}
\]

Simpson’s rule is considered the most accurate of the approximate methods. An answer of 4.5 square units can be achieved with the other two methods if more intervals are taken.

2. Plot the graph of \( y = 2x^2 + 3 \) between \( x = 0 \) and \( x = 4 \). Estimate the area enclosed by the curve, the ordinates \( x = 0 \) and \( x = 4 \), and the \( x \)-axis by an approximate method.

A table of values is shown below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 0.5 )</th>
<th>( 1.0 )</th>
<th>( 1.5 )</th>
<th>( 2.0 )</th>
<th>( 2.5 )</th>
<th>( 3.0 )</th>
<th>( 3.5 )</th>
<th>( 4.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 2x^2 + 3 )</td>
<td>3</td>
<td>3.5</td>
<td>5.0</td>
<td>7.5</td>
<td>11.0</td>
<td>15.5</td>
<td>21.0</td>
<td>27.5</td>
<td>35.0</td>
</tr>
</tbody>
</table>

A graph of \( y = 2x^2 + 3 \) is shown below.
Using Simpson’s rule with 8 intervals each of width 0.5 gives:

\[
\text{area} \approx \frac{1}{3} \cdot 0.5 \left[ (3 + 35) + 4(3.5 + 7.5 + 15.5 + 27.5) + 2(5.0 + 11.0 + 21.0) \right]
\]

\[
= \frac{1}{3} \cdot 0.5 \left[ 38 + 4(54) + 2(37) \right] = \frac{1}{3} \cdot 0.5 \left[ 38 + 216 + 74 \right]
\]

\[
= \frac{1}{3} \cdot 0.5 \cdot 328 = 54.7 \text{ square units}
\]

3. The velocity of a car at one-second intervals is given in the following table:

<table>
<thead>
<tr>
<th>time ( t ) (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity ( v ) (m/s)</td>
<td>0</td>
<td>2.0</td>
<td>4.5</td>
<td>8.0</td>
<td>14.0</td>
<td>21.0</td>
<td>29.0</td>
</tr>
</tbody>
</table>

Determine the distance travelled in six seconds (i.e. the area under the \( v/t \) graph) using Simpson’s rule.

Using Simpson’s rule with six intervals each of width 1 s gives:

\[
\text{distance} \approx \frac{1}{3} \cdot 1 \left[ (0 + 29.0) + 4(2.0 + 8.0 + 21.0) + 2(4.5 + 14.0) \right] = \frac{1}{3} \left[ 29.0 + 124.0 + 37.0 \right]
\]

\[
= \frac{1}{3} \cdot 190 = 63.33 \text{ m}
\]

4. The shape of a piece of land is shown below. To estimate the area of the land, a surveyor takes measurements at intervals of 50 m, perpendicular to the straight portion with the results shown (the dimensions being in metres). Estimate the area of the land in hectares (1 ha = \( 10^4 \) m\(^2\)).

Using Simpson’s rule with six intervals each of width 50 m gives:

\[
\text{area} \approx \frac{1}{3} \cdot 50 \left[ (140 + 0) + 4(160 + 190 + 130) + 2(200 + 180) \right] = \frac{1}{3} \cdot 50 \left[ 140 + 1920 + 760 \right]
\]
5. The deck of a ship is 35 m long. At equal intervals of 5 m the width is given by the following table:

<table>
<thead>
<tr>
<th>Width (m)</th>
<th>0</th>
<th>2.8</th>
<th>5.2</th>
<th>6.5</th>
<th>5.8</th>
<th>4.1</th>
<th>3.0</th>
<th>2.3</th>
</tr>
</thead>
</table>

Estimate the area of the deck.

Using the trapezoidal rule with seven intervals each of width 5 m gives:

\[ \text{area} \approx (5) \left( \frac{0 + 2.3}{2} + 2.8 + 5.2 + 6.5 + 5.8 + 4.1 + 3.0 \right) = (5)[1.15 + 27.4] \]

\[ = (5)(28.55) = 143 \text{ m}^2 \]

(To use Simpson’s rule needs an even number of intervals, so could not be used in this question.)
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1. The areas of equidistantly spaced sections of the underwater form of a small boat are as follows:

\[ 1.76, \ 2.78, \ 3.10, \ 3.12, \ 2.61, \ 1.24, \ 0.85 \text{ m}^2 \]

Determine the underwater volume if the sections are 3 m apart.

Underwater volume \( = \frac{3}{3} \left[ (1.76 + 0.85) + 4(2.78 + 3.12 + 1.24) + 2(3.10 + 2.61) \right] \)

\[ = 2.61 + 28.56 + 11.42 = 42.59 \text{ m}^3 \]

2. To estimate the amount of earth to be removed when constructing a cutting, the cross-sectional area at intervals of 8 m were estimated as follows:

\[ 0, \ 2.8, \ 3.7, \ 4.5, \ 4.1, \ 2.6, \ 0 \text{ m}^3 \]

Estimate the volume of earth to be excavated.

Volume of earth to be excavated \( = \frac{8}{3} \left[ (0 + 0) + 4(2.8 + 4.5 + 2.6) + 2(3.7 + 4.1) \right] \)

\[ = \frac{8}{3} \left[ 39.6 + 15.6 \right] = 147 \text{ m}^3 \text{ (correct to 3 significant figures)} \]

3. The circumference of a 12 m long log of timber of varying circular cross-section is measured at intervals of 2 m along its length and the results are:

<table>
<thead>
<tr>
<th>Distance from one end (m)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference (m)</td>
<td>2.80</td>
<td>3.25</td>
<td>3.94</td>
<td>4.32</td>
<td>5.16</td>
<td>5.82</td>
<td>6.36</td>
</tr>
</tbody>
</table>

Estimate the volume of the timber in cubic metres.

If circumference \( c = 2\pi r \) then \( r = \frac{c}{2\pi} \)

Cross-sectional area \( = \pi r^2 = \pi \left( \frac{c}{2\pi} \right)^2 = \frac{c^2}{4\pi} \)

Hence, the cross-sectional areas are:

\[ \frac{2.80^2}{4\pi} = 0.6239 \text{ m}^2, \quad \frac{3.25^2}{4\pi} = 0.8405 \text{ m}^2, \quad 1.2353 \text{ m}^2, \]
Hence, volume of timber

\[ \approx \frac{2}{3} \left[ (0.6239 + 3.2189) + 4(0.8405 + 1.4851 + 2.6955) + 2(1.2353 + 2.1188) \right] \]

\[ = \frac{2}{3} \left( 3.8428 + 20.0844 + 6.7082 \right) = \frac{2}{3} \left( 30.6354 \right) \]

\[ = 20.42 \, \text{m}^3 \]
1. Determine the mean value of the periodic waveforms shown below over a half cycle.

(a) Over half a cycle, mean value = \[
\text{area under curve} \div \text{length of base} = \left(\frac{2 \times 10^{-3}}{1 \times 10^{-3}}\right) \times \frac{10}{10} \times 10^3 = 2 \text{ A}
\]

(b) Over half a cycle, mean value = \[
\frac{1}{2} \left(\frac{5 \times 10^{-3}}{5 \times 10^{-3}}\right) \times 100 = 50 \text{ V}
\]

(c) Over half a cycle, mean value = \[
\frac{1}{2} \left(\frac{15 \times 10^{-3}}{15 \times 10^{-3}}\right) \times 5 = 2.5 \text{ A}
\]

2. Find the average value of the periodic waveforms shown below over one complete cycle

(a) Mean value = \[
\text{area under curve} \div \text{length of base} = \left(\frac{1}{2} \times 2 \times 10^{-3} \times 10 \times 10^{-3}\right) \div 4 \times 10^{-3} = 2.5 \text{ mV}
\]

(b) Mean value = \[
\text{area under curve} \div \text{length of base} = \left(\frac{1}{2} \times (2 + 4)(5) \times 10^{-3}\right) \div 5 \times 10^{-3} = 3 \text{ A}
\]
3. An alternating current has the following values at equal intervals of 5 ms:

<table>
<thead>
<tr>
<th>Time (ms)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current (A)</td>
<td>0</td>
<td>0.9</td>
<td>2.6</td>
<td>4.9</td>
<td>5.8</td>
<td>3.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Plot a graph of current against time and estimate the area under the curve over the 30 ms period using the mid-ordinate rule and determine its mean value.

A graph of current against time is shown plotted below

Mid-ordinates are shown by the broken lines in the above diagram. The mid-ordinate values are:

0.4, 1.6, 3.8, 5.7, 4.9 and 2.2

\[
\text{area} \approx (5 \times 10^{-3})[0.4+1.6+3.8+5.7+4.9+2.2] \\
= (5 \times 10^{-3})[18.6] = 93 \times 10^{-3} \text{ As} = 0.093 \text{ As}
\]

Mean value = \[
\frac{\text{area}}{\text{length of base}} = \frac{93 \times 10^{-3} \text{ As}}{30 \times 10^{-3} \text{s}} = 3.1 \text{ A}
\]

4. Determine, using an approximate method, the average value of a sine wave of maximum value 50 V for (a) a half cycle and (b) a complete cycle.

Let \( y = 50 \sin \theta \)

(a) Taking 30° intervals:
\[ \theta \quad 0 \quad \pi/6 (30^\circ) \quad \pi/3 (60^\circ) \quad \pi/2 (90^\circ) \quad 2\pi/3 (120^\circ) \quad 5\pi/6 (150^\circ) \quad \pi (180^\circ) \]

\[ y = 50 \sin \theta \quad 0 \quad 25 \quad 43.3 \quad 50 \quad 43.3 \quad 25 \quad 0 \]

Using Simpson’s rule with six intervals each of width \( \pi/6 \) gives:

\[
\text{area} \approx \frac{1}{3}(\pi/6) \left[ (0 + 0) + 4(25 + 50 + 25) + 2(43.3 + 43.3) \right] = \frac{1}{3}(\pi/6)[0 + 400 + 173.2] = \frac{1}{3}(\pi/6)(573.2) = 100 \text{ square units}
\]

Hence, average value \( \approx \frac{100}{\pi} \approx 31.83 \text{ V} \)

(b) The average value of a sine wave over a complete cycle is zero (area above horizontal axis is equal to the area below)

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5. An indicator diagram of a steam engine is 12 cm long. Seven evenly spaced ordinates, including the end ordinates, are measured as follows:

\[ 5.90, \quad 5.52, \quad 4.22, \quad 3.63, \quad 3.32, \quad 3.24, \quad 3.16 \text{ cm} \]

Determine the area of the diagram and the mean pressure in the cylinder if 1 cm represents 90 kPa.

Area \( \approx \frac{1}{3}(2)\left[ (5.90 + 3.16) + 4(5.52 + 3.63 + 3.24) + 2(4.22 + 3.32) \right] \]

\[
= \frac{1}{3}(2)[9.06 + 49.56 + 15.08] = \frac{1}{3}(2)(73.7) = 49.13 \text{ cm}^2
\]

Mean value \( = \frac{49.13 \text{ cm}^2}{12 \text{ cm}} \times 90 \times 10^3 \text{ Pa/cm} = 368.5 \text{ kPa} \)