CHAPTER 18 POLYNOMIAL DIVISION AND THE FACTOR AND REMAINDER THEOREMS

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1. Divide \((2x^2 + xy - y^2)\) by \((x + y)\)

\[
\begin{align*}
2x - y \\
\overline{x + y) 2x^2 + xy - y^2} \\
2x^2 + 2xy \\
- xy - y^2 \\
- xy - y^2 \\
\end{align*}
\]

Hence, \(\frac{2x^2 + xy - y^2}{x + y} = 2x - y\)

2. Divide \((3x^2 + 5x - 2)\) by \((x + 2)\)

\[
\begin{align*}
3x - 1 \\
\overline{x + 2) 3x^2 + 5x - 2} \\
3x^2 + 6x \\
- x - 2 \\
- x - 2 \\
\end{align*}
\]

Hence, \(\frac{3x^2 + 5x - y^2}{x + 2} = 3x - 1\)

3. Determine \((10x^2 + 11x - 6)\) \(\div\) \((2x + 3)\)

\[
\begin{align*}
5x - 2 \\
\overline{2x + 3) 10x^2 + 11x - 6} \\
10x^2 + 15x \\
- 4x - 6 \\
- 4x - 6 \\
\end{align*}
\]

Hence, \(\frac{10x^2 + 11x - 6}{2x + 3} = 5x - 2\)
4. Find \( \frac{14x^2 - 19x - 3}{2x - 3} \)

\[
\begin{align*}
7x + 1 \\
2x - 3) & \quad 14x^2 - 19x - 3 \\
& \quad 14x^2 - 21x \\
& \quad 2x - 3 \\
& \quad 2x - 3
\end{align*}
\]

Hence, \( \frac{14x^2 - 19x - 3}{2x - 3} = 7x + 1 \)

5. Divide \((x^3 + 3x^2y + 3xy^2 + y^3)\) by \((x + y)\)

\[
\begin{align*}
x^2 + 2xy + y^2 \\
x + y) & \quad x^3 + 3x^2y + 3xy^2 + y^3 \\
& \quad x^3 + x^2y \\
& \quad 2x^2y + 3xy^2 \\
& \quad 2x^2y + 2xy^2 \\
& \quad xy^2 + y^3 \\
& \quad xy^2 + y^3
\end{align*}
\]

Hence, \( \frac{x^3 + 3x^2y + 3xy^2 + y^3}{x + y} = x^2 + 2xy + y^2 \)

6. Find \((5x^2 - x + 4) \div (x - 1)\)

\[
\begin{align*}
5x + 4 \\
x - 1) & \quad 5x^2 - x + 4 \\
& \quad 5x^2 - 5x \\
& \quad 4x + 4 \\
& \quad 4x - 4 \\
& \quad 8
\end{align*}
\]

Hence, \( \frac{5x^2 - x + 4}{x - 1} = 5x + 4 + \frac{8}{x - 1} \)

7. Divide \((3x^3 + 2x^2 - 5x + 4)\) by \((x + 2)\)

\[
\begin{align*}
3x^2 - 4x + 3 \\
x + 2) & \quad 3x^3 + 2x^2 - 5x + 4
\end{align*}
\]
\[
3x^3 + 6x^2 \\
-4x^2 - 5x \\
-4x^2 - 8x \\
3x + 4 \\
3x + 6 \\
-2
\]

Hence, \( \frac{3x^3 + 2x^2 - 5x + 4}{x + 2} = 3x^2 - 4x + 3 - \frac{2}{x + 2} \)

8. Determine \( \frac{5x^4 + 3x^3 - 2x + 1}{x - 3} \)

\[
\begin{align*}
5x^4 + 18x^2 + 54x + 160 \\
x - 3 \frac{5x^4 + 3x^3 - 2x + 1}{5x^4 - 15x^3} \\
18x^3 \\
18x^3 - 54x^2 \\
54x^2 - 2x \\
54x^2 - 162x \\
160x + 1 \\
160x - 480 \\
481
\end{align*}
\]

Hence, \( \frac{5x^4 + 3x^3 - 2x + 1}{x - 3} = 5x^3 + 18x^2 + 54x + 160 + \frac{481}{x - 3} \)
1. Use the factor theorem to factorize: $x^2 + 2x - 3$

Let $f(x) = x^2 + 2x - 3$

If $x = 1$, $f(x) = 1 + 2 - 3 = 0$ hence, $(x - 1)$ is a factor

$x = 2$, $f(x) = 4 + 4 - 3 = 5$

$x = 3$, $f(x) = 9 + 6 - 3 = 12$

$x = -1$, $f(x) = 1 - 2 - 3 = -4$

$x = -2$, $f(x) = 4 - 4 - 3 = -3$

$x = -3$, $f(x) = 9 - 6 - 3 = 0$ hence, $(x + 3)$ is a factor

Thus, $x^2 + 2x - 3 = (x - 1)(x + 3)$

2. Use the factor theorem to factorize: $x^3 + x^2 - 4x - 4$

Let $f(x) = x^3 + x^2 - 4x - 4$

If $x = 1$, $f(x) = 1 + 1 - 4 - 4 = -6$

$x = 2$, $f(x) = 8 + 4 - 8 - 4 = 0$ hence, $(x - 2)$ is a factor

$x = 3$, $f(x) = 27 + 9 - 12 - 4 = 20$

$x = -1$, $f(x) = -1 + 1 + 4 - 4 = 0$ hence, $(x + 1)$ is a factor

$x = -2$, $f(x) = -8 + 4 + 8 - 4 = 0$ hence, $(x + 2)$ is a factor

Thus, $x^3 + x^2 - 4x - 4 = (x + 1)(x + 2)(x - 2)$

3. Use the factor theorem to factorize: $2x^3 + 5x^2 - 4x - 7$

Let $f(x) = 2x^3 + 5x^2 - 4x - 7$

If $x = 1$, $f(x) = 2 + 5 - 4 - 7 = -4$

$x = 2$, $f(x) = 16 + 20 - 8 - 7 = 21$

$x = 3$, $f(x) = 45 + 12 - 7 = 50$
\[ x = -1, f(x) = -2 + 5 + 4 - 7 = 0 \text{ hence, } (x + 1) \text{ is a factor} \]
\[ x = -2, f(x) = -16 + 20 + 8 - 7 = 5 \]
\[ x = -3, f(x) = -54 + 45 + 12 - 7 = -4 \]

Since the first term dominates, there are no further factors.

\[
\begin{align*}
2x^2 + 3x - 7 \\
\underline{x + 1\overline{2x^3 + 5x^2 - 4x - 7}} \\
x + 1
\end{align*}
\]

\[
\begin{align*}
2x^3 + 5x^2 - 4x - 7 &= \overline{2x^2 + 2x^2} \\
&\overline{3x^2 - 4x} \\
&\overline{3x^2 + 3x} \\
&\overline{-7x - 7} \\
&\overline{-7x - 7}
\end{align*}
\]

Thus, \[ 2x^3 + 5x^2 - 4x - 7 = (x + 1)(2x^2 + 3x - 7) \]

4. Use the factor theorem to factorize: \[ 2x^3 - x^2 - 16x + 15 \]

Let \(f(x) = 2x^3 - x^2 - 16x + 15\)

If \(x = 1\), \(f(x) = 2 - 1 - 16 + 15 = 0 \text{ hence, } (x - 1) \text{ is a factor}\)
\[ x = 2, f(x) = 16 - 4 - 32 + 15 = -5 \]
\[ x = 3, f(x) = 54 - 9 - 48 + 15 = 12 \]
\[ x = -1, f(x) = -1 - 1 + 16 + 15 = 29 \]
\[ x = -2, f(x) = -16 - 4 + 32 + 15 = 27 \]
\[ x = -3, f(x) = -54 - 9 + 48 + 15 = 0 \text{ hence, } (x + 3) \text{ is a factor}\]

\[
\begin{align*}
2x^3 - x^2 - 16x + 15 &= \frac{2x^3 - x^2 - 16x + 15}{(x-1)(x+3)} \\
\end{align*}
\]

\[
\begin{align*}
2x - 5 \\
x^2 + 2x - 3 \overline{2x^3 - x^2 - 16x + 15} \\
2x^3 + 4x^2 - 6x \\
-5x^2 - 10x + 15 \\
-5x^2 - 10x + 15
\end{align*}
\]

Hence, \[ 2x^3 - x^2 - 16x + 15 = (x - 1)(x + 3)(2x - 5) \]
5. Use the factor theorem to factorize \( x^3 + 4x^2 + x - 6 \) and hence solve the cubic equation \( x^3 + 4x^2 - x - 6 = 0 \)

Let \( f(x) = x^3 + 4x^2 - x - 6 \)

If \( x = 1 \), \( f(x) = 1 + 4 + 1 - 6 = 0 \) hence, \((x - 1)\) is a factor

\[ x = 2, \quad f(x) = 8 + 16 - 2 - 6 = 16 \]
\[ x = -1, \quad f(x) = -1 + 4 - 1 - 6 = -4 \]
\[ x = -2, \quad f(x) = -8 + 16 - 2 - 6 = 0 \] hence, \((x + 2)\) is a factor

\[ x = -3, \quad f(x) = -27 + 36 - 3 - 6 = 0 \] hence, \((x + 3)\) is a factor

Thus, \( x^3 + 4x^2 + x - 6 = (x - 1)(x + 2)(x + 3) \)

If \( x^3 + 4x^2 - x - 6 = 0 \) then \((x - 1)(x + 2)(x + 3) = 0\)

from which, \( x = 1, x = -2 \) or \( x = -3 \)

6. Solve the equation: \( x^3 - 2x^2 - x + 2 = 0 \)

Let \( f(x) = x^3 - 2x^2 - x + 2 \)

If \( x = 1 \), \( f(x) = 1 - 2 - 1 + 2 = 0 \) hence, \((x - 1)\) is a factor

\[ x = 2, \quad f(x) = 8 - 8 - 2 + 2 = 0 \] hence, \((x - 2)\) is a factor
\[ x = 3, \quad f(x) = 27 - 18 - 3 + 2 = 8 \]
\[ x = -1, \quad f(x) = -1 - 2 + 1 + 2 = 0 \] hence, \((x + 1)\) is a factor

Hence, \( x^3 - 2x^2 - x + 2 = (x - 1)(x - 2)(x + 1) \)

If \( x^3 - 2x^2 - x + 2 = 0 \) then \((x - 1)(x - 2)(x + 1) = 0\)

from which, \( x = 1, x = 2, \) or \( x = -1 \)
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1. Find the remainder when \(3x^2 - 4x + 2\) is divided by
   (a) \((x - 2)\)  (b) \((x + 1)\)

   (a) Remainder is \(ap^2 + bp + c\) where \(a = 3, b = -4, c = 2\) and \(p = 2\)
   i.e. the remainder is: \(3(2)^2 + (-4)(2) + 2 = 12 - 8 + 2 = 6\)

   (b) Remainder is \(ap^2 + bp + c\) where \(a = 3, b = -4, c = 2\) and \(p = -1\)
   i.e. the remainder is: \(3(-1)^2 + (-4)(-1) + 2 = 3 + 4 + 2 = 9\)

2. Determine the remainder when \(x^3 - 6x^2 + x - 5\) is divided by
   (a) \((x + 2)\)  (b) \((x - 3)\)

   (a) Remainder is \(ap^3 + bp^2 + cp + d\) where \(a = 1, b = -6, c = 1, d = -5\) and \(p = -2\)
   Hence, remainder = \(1(-2)^3 - 6(-2)^2 + 1(-2) - 5 = -8 - 24 - 2 - 5 = -39\)

   (b) When \(p = 3\), remainder = \(1(3)^3 - 6(3)^2 + 1(3) - 5 = 27 - 54 + 3 - 5 = -29\)

3. Use the remainder theorem to find the factors of \(x^3 - 6x^2 + 11x - 6\)

   Remainder is \(ap^3 + bp^2 + cp + d\) where \(a = 1, b = -6, c = 11, d = -5\) and \(p = a\).

   If a value for \(p\) is chosen which makes the remainder zero, then a factor \((x - a)\) would exist.

   If \(p = 1\), then remainder = \(1(1)^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 0\); hence, \((x - 1)\) is a factor

   If \(p = 2\), then remainder = \(1(2)^3 - 6(2)^2 + 11(2) - 6 = 8 - 24 + 22 - 6 = 0\); hence, \((x - 2)\) is a factor

   If \(p = 3\), then remainder = \(1(3)^3 - 6(3)^2 + 11(3) - 6 = 27 - 54 + 33 - 6 = 0\); hence, \((x - 3)\) is a factor

   Hence, \(x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)\)

4. Determine the factors of \(x^3 + 7x^2 + 14x + 8\) and hence solve the cubic equation

   \(x^3 + 7x^2 + 14x + 8 = 0\)
Remainder is $ap^3 + bp^2 + cp + d$ where $a = 1$, $b = 7$, $c = 14$, $d = 8$

Let $p = 1$, then remainder = $1(1)^3 + 7(1)^2 + 14(1) + 8 = 30$

Let $p = -1$, then remainder = $1(-1)^3 + 7(-1)^2 + 14(-1) + 8 = -1 + 7 - 14 + 8 = 0$, hence $(x + 1)$ is a factor

Let $p = -2$, then remainder = $1(-2)^3 + 7(-2)^2 + 14(-2) + 8 = -8 + 28 - 28 + 8 = 0$, hence $(x + 2)$ is a factor

Let $p = -3$, then remainder = $1(-3)^3 + 7(-3)^2 + 14(-3) + 8 = -27 + 63 - 42 + 8 = 2$

Let $p = -4$, then remainder = $1(-4)^3 + 7(-4)^2 + 14(-4) + 8 = -64 + 112 - 56 + 8 = 0$, hence $(x + 4)$ is a factor

Hence, $x^3 + 7x^2 + 14x + 8 = (x + 1)(x + 2)(x + 4)$

If $x^3 + 7x^2 + 14x + 8 = 0$ then $(x + 1)(x + 2)(x + 4) = 0$

from which, $x = -1$, $x = -2$ or $x = -4$

5. Determine the value of $a$ if $(x + 2)$ is a factor of $(x^3 – ax^2 + 7x + 10)$

If $(x + 2)$ is a factor then $x = -2$

Hence, $(-2)^3 - a(-2)^2 + 7(-2) + 10 = 0$

i.e. $-8 - 4a - 14 + 10 = 0$

from which, $-8 - 14 + 10 = 4a$

i.e. $4a = -12$

and $a = -12/4 = -3$

6. Using the remainder theorem, solve the equation $2x^3 – x^2 – 7x + 6 = 0$

Remainder is $ap^3 + bp^2 + cp + d$ where $a = 2$, $b = -1$, $c = -7$, $d = 6$

Let $p = 1$, then remainder = $2(1)^3 + (-1)(1)^2 + (-7)(1) + 6 = 2 - 1 - 7 + 6 = 0$, hence $(x - 1)$ is a factor

Let $p = 2$, then remainder = $2(2)^3 + (-1)(2)^2 + (-7)(2) + 6 = 16 - 4 - 14 + 6 = 4$
Let $p = -1$, then remainder $= 2(-1)^3 + (-1)(-1)^2 + (-7)(-1) + 6 = -2 - 1 + 7 + 6 = 10$

Let $p = -2$, then remainder $= 2(-2)^3 + (-1)(-2)^2 + (-7)(-2) + 6 = -16 - 4 + 14 + 6 = 0$, hence $(x + 2)$ is a factor

Let $p = -3$, then remainder $= 2(-3)^3 + (-1)(-3)^2 + (-7)(-3) + 6 = -54 - 9 + 21 + 6 = -36$

The third root can be found by division, i.e.

$$\frac{2x^3 - x^2 - 7x + 6}{(x-1)(x+2)} = \frac{2x^3 - x^2 - 7x + 6}{x^2 + x - 2}$$

$$= \frac{2x - 3}{x^2 + x - 2} 2x^3 - x^2 - 7x + 6$$

$$= \frac{2x^3 + 2x^2 - 4x}{2x^3 - 7x + 6}$$

$$= \frac{-3x^2 - 3x + 6}{-3x^2 - 3x + 6}$$

Hence, $2x^3 - x^2 - 7x + 6 = (x - 1)(x + 2)(2x - 3)$

If $2x^3 - x^2 - 7x + 6 = 0$ then $(x - 1)(x + 2)(2x - 3) = 0$

from which, $x = 1$, $x = -2$ or $x = 1.5$